

INVERSE SEMIGROUP EQUIVARIANT KK -THEORY AND C^* -EXTENSIONS

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ABSTRACT. In this note we extend the classical result by G. G. Kasparov that the Kasparov groups $KK_1(A, B)$ can be identified with the extension groups $\text{Ext}(A, B)$ to the inverse semigroup equivariant setting. More precisely, we show that $KK_G^1(A, B) \cong \text{Ext}_G(A \otimes \mathcal{K}_G, B \otimes \mathcal{K}_G)$ for every countable, E -continuous inverse semigroup G . For locally compact second countable groups G this was proved by K. Thomsen, and technically this note presents an adaptation of his proof.

1.

In Theorem 7.1 of [6], G. G. Kasparov shows that for every *compact* second countable group G and ungraded separable G -algebras A and B (where A is nuclear) there exists an isomorphism between the extension group $\text{Ext}(A, B)$ and the Kasparov group $KK^1(A, B) := KK(A, B \oplus B)$. In [11], K. Thomsen generalizes this result to locally compact second countable Hausdorff groups G and ungraded separable G -algebras A and B by establishing an isomorphism between $\text{Ext}_G(A \otimes \mathcal{K}_G, B \otimes \mathcal{K}_G)$ and $KK_G^1(A, B)$, where $\mathcal{K}_G = \mathcal{K} \otimes \mathcal{K}(L^2(G))$. Very roughly speaking the proof is done by considering Kasparov's proof for these more general groups G . Some unitaries which fall out due to G -equivariance during the process of equivalently transforming Kasparov cycles to another format called KK -cocycles are not averaged away, but kept as unitary cocycles and then transferred into the algebra \mathcal{K}_G by some equivariance theorems by J. A. Mingo and W. J. Phillips [9].

In this note we prove the analogous result for countable, E -continuous inverse semigroups G by adapting K. Thomsen's proof.

We will here not repeat Thomsen's complete proof but only show the modifications that have to be made when reading Thomsen's paper for an inverse semigroup G . We will be mainly focused on details, leaving the greater context to the original paper by Thomsen. The only thing we have to observe and to take care of is the G -action. The definitions of equivariant KK -theory for groups and inverse semigroups are very close, see Section 2, but the difference that $gg^{-1} = 1$ in a group but gg^{-1} is just a projection in an inverse semigroup, is unavoidable. The short answer of this note is that we shall make only the following modifications in Thomsen's paper [11]:

- If one side of an identity contains an action α_g but the other side does not, then we have to add $\alpha_{gg^{-1}}$ to the other side. For example, $\alpha_g(x) = y$

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has to be changed to $\alpha_g(x) = \alpha_{gg^{-1}}(y)$. This is obvious since α_g loses information by projecting onto $\alpha_{gg^{-1}}$.

- A unitary cocycle has to be replaced by a cocycle in the sense of Definition 4.1. It is obvious that we cannot work with unitaries. The unitary operator definition changes to a partial isometry definition with identical source and range projection.
- The $\ell^2(G)$ -space has to be completely replaced by a compatible $\ell^2(G)$ -space, see Definition 8.4. Without a suitable change the algebra $\mathcal{K}(\ell^2(G))$ were no longer a G -algebra and the proof would break down. The construction of this compatible $\ell^2(G)$ -space requires however the inverse semigroup G to be *E-continuous*, see Definition 8.2.

More or less we could finish here our note, but we shall nevertheless discuss some of the modifications in detail to convince the reader that everything goes through.

An important help, and a fact which is used again and again, is that the operator $\alpha_{gg^{-1}} \in \mathcal{ZM}(A)$ is always in the center of the multiplier algebra of a given G -algebra (A, α) . This includes also the G -algebra of adjoint-able operators $\mathcal{L}(\mathcal{E})$ of a given G -Hilbert module \mathcal{E} . In Thomsen's proof $\alpha_{gg^{-1}} = 1$ and so $\alpha_{gg^{-1}}$ is absorbed by any neighboring element via multiplication. In our proof $\alpha_{gg^{-1}}$ often cannot vanish in its position, but since it is in the center, can move around within an expression until it is absorbed elsewhere, for example another presence of α_g through the identity $\alpha_{gg^{-1}}\alpha_g = \alpha_g$. With that technique, Thomsen's proof can be kept under the weaker assumptions.

A different approach in generalizing Kasparov's result to locally compact groups is given by J. Cuntz [4].

The structure of this note is as follows. The Sections 2 to 7 represent corresponding sections of Thomsen's paper, with the same numbers and titles. There we discuss in each section the modifications that have to be performed in the corresponding section in [11]. Section 2 includes also a brief summary of inverse semigroup equivariant KK -theory, and Section 7 contains the above mentioned main result in Theorems 7.1 and 7.2. The last Section 8 presents an appendix where we introduce the compatible $\ell^2(G)$ -space and adapt certain triviality results from Mingo and Phillips [9] for this $\ell^2(G)$ -space.

2. THE BUSBY-INVARIANT IN THE EQUIVARIANT CASE

Let G denote a countable inverse semigroup. We shall denote the involution on G by $g \mapsto g^{-1}$ (determined by $gg^{-1}g = g$). A semigroup homomorphism is said to be *unital* if it preserves the identity $1 \in G$ and the zero element $0 \in G$, provided that the involved inverse semigroups are gifted with any of these elements. In this note, we shall however require none of them. We consider G -equivariant KK -theory as defined in [3] but make a slight adaption by making this theory *compatible* in the sense that we only allow for *compatible* Hilbert (bi)modules (cf. Definition 2.2). We are going to recall the basic definitions of KK^G .

Definition 2.1. A G -algebra (A, α) is a $\mathbb{Z}/2$ -graded C^* -algebra A with a unital semigroup homomorphism $\alpha : G \rightarrow \text{End}(A)$ such that α_g respects the grading and $\alpha_{gg^{-1}}(x)y = x\alpha_{gg^{-1}}(y)$ for all $x, y \in A$ and $g \in G$.

Definition 2.2. A G -Hilbert B -module \mathcal{E} is a $\mathbb{Z}/2$ -graded Hilbert module over a G -algebra (B, β) endowed with a unital semigroup homomorphism $G \rightarrow \text{Lin}(\mathcal{E})$ (linear maps on \mathcal{E}) such that U_g respects the grading and $\langle U_g(\xi), U_g(\eta) \rangle = \beta_g(\langle \xi, \eta \rangle)$, $U_g(\xi b) = U_g(\xi)\beta_g(b)$, and $U_{gg^{-1}}(\xi)b = \xi\beta_{gg^{-1}}(b)$ for all $g \in G, \xi, \eta \in \mathcal{E}$ and $b \in B$.

In the last definition, $U_{gg^{-1}}$ is automatically a self-adjoint projection in the center of $\mathcal{L}(\mathcal{E})$ (because adjointable operators are B -linear and $U_{gg^{-1}}(\xi)b = \xi\beta_{gg^{-1}}(b)$), and the G -action $G \rightarrow \text{End}(\mathcal{L}(\mathcal{E}))$ given by $g(T) = U_g T U_{g^{-1}}$ turns $\mathcal{L}(\mathcal{E})$ to a G -algebra ($g \in G$ and $T \in \mathcal{L}(\mathcal{E})$). A G -algebra (A, α) is a G -Hilbert module over itself under the inner product $\langle a, b \rangle = a^*b$ and $U := \beta := \alpha$ in the last definition. A $*$ -homomorphism between G -algebras is called G -equivariant if it intertwines the G -action. A G -Hilbert A, B -bimodule over G -algebras A and B is a G -Hilbert B -module \mathcal{E} equipped with a G -equivariant $*$ -homomorphism $A \rightarrow \mathcal{L}(\mathcal{E})$. The compact operators on a separable Hilbert space are written as \mathcal{K} , $\mathcal{K}(\mathcal{E}) \subseteq \mathcal{L}(\mathcal{E})$ denotes the compact operators on a Hilbert module \mathcal{E} , and $\theta_{x,y} \in \mathcal{K}(\mathcal{E})$ the elementary compact operators $\theta_{x,y}(z) := x\langle y, z \rangle$ for all $x, y, z \in \mathcal{E}$.

Definition 2.3. Let A and B be G -algebras. We define a Kasparov cycle (\mathcal{E}, T) , where \mathcal{E} is a G -Hilbert A, B -bimodule, to be an ordinary Kasparov cycle (without G -action) (see [6, 7]) satisfying $U_g T U_{g^{-1}} - T U_{gg^{-1}} \in \{S \in \mathcal{L}(\mathcal{E}) \mid aS, Sa \in \mathcal{K}(\mathcal{E}) \text{ for all } a \in A\}$ for all $g \in G$. The Kasparov group $KK^G(A, B)$ is defined to be the collection $\mathbb{E}^G(A, B)$ of these cycles divided by homotopy induced by $\mathbb{E}^G(A, B[0, 1])$.

There exists an associative Kasparov product in KK^G as usual (see [3]).

We list here some notions from [11]. One is given two ungraded separable G -algebras (A, α) and (B, β) . The G -algebra B is assumed to be weakly stable, that is, there exists a G -equivariant isomorphism $(B, \beta) \cong (B \otimes \mathcal{K}, \beta \otimes id_{\mathcal{K}})$. The multiplier and corona algebra of A is denoted by $\mathcal{M}(A)$ and $Q(A) := \mathcal{M}(A)/A$, respectively.

We remark that the identity $\mathcal{M}(A) \cong \mathcal{L}_A(A)$ for a C^* -algebra A is often and freely used. (We recall that the isomorphism is given by mapping an operator $T \in \mathcal{L}_A(A)$ to the double centralizer $(T, T') \in \mathcal{M}(A)$, where $T'(a) := (T^*(a^*))^*$ for all $a \in A$.) In particular, $\mathcal{M}(B \otimes \mathcal{K})$ and $\mathcal{L}_{B \otimes \mathcal{K}}(B \otimes \mathcal{K})$ will be often identified.

Definition 2.4. By using the $*$ -isomorphism $\mathcal{M}(A) \cong \mathcal{L}(A)$, the multiplier algebra turns to a G -algebra $(\mathcal{M}(A), \bar{\alpha}) \cong (\mathcal{L}(A), \bar{\alpha})$ under the G -action $\bar{\alpha} : G \rightarrow \text{End}(\mathcal{L}(A))$ determined by $\bar{\alpha}_g(T) := \alpha_g \circ T \circ \alpha_{g^{-1}}$ for all $g \in G$ and $T \in \mathcal{L}(A)$ as in Kasparov [5, 1.4].

Thomsen interprets $\bar{\alpha}_g$ as the strictly continuous extension of α_g from A to $\mathcal{M}(A)$. We shall however always use the aforementioned explicit formula in our computations. This G -action on the multiplier algebra induces a canonical G -action $\hat{\alpha} : G \rightarrow \text{End}(Q(A))$ on the corona algebra, which turns it to a G -algebra $(Q(A), \hat{\alpha})$. The G -equivariant quotient map between multiplier and corona algebra is denoted by $q_A : \mathcal{M}(A) \rightarrow Q(A)$. A G -equivariant $*$ -homomorphism $\varphi : A \rightarrow B$ which is quasi-unital (i.e. $\overline{\text{span}} \varphi(A)B = pB$ for some projection $p \in \mathcal{M}(B)$) induces a well-known strictly continuous $*$ -homomorphism $\bar{\varphi} : \mathcal{M}(A) \rightarrow \mathcal{M}(B)$ between the multiplier algebras, which once again induces a canonical G -equivariant $*$ -homomorphism $\hat{\varphi} : Q(A) \rightarrow Q(B)$.

Definition 2.5. An operator $T \in \mathcal{L}(\mathcal{E})$ on a G -Hilbert module \mathcal{E} with G -action U is called G -invariant if $T \circ U_g = U_g \circ T$ (equivalently, $U_g T U_{g^{-1}} = T U_g U_{g^{-1}}$ or $U_g T U_{g^{-1}} = U_{gg^{-1}} T U_{gg^{-1}}$) for all $g \in G$.

In case of a multiplier algebra, G -equivariance is also called as follows.

Definition 2.6. An operator in $T \in \mathcal{M}((B, \beta))$ is called $\overline{\beta}$ -invariant if T (and consequently T^*) commutes with β_g for all $g \in G$ (equivalently: $\overline{\beta}_g(T) = T \overline{\beta}_g(1)$ or $\overline{\beta}_g(T) = \overline{\beta}_{gg^{-1}}(T)$ for all $g \in G$).

The operators $V_1, V_2 \in \mathcal{M}((B, \beta))$ always denote $\overline{\beta}$ -invariant isometries such that $V_1 V_1^* + V_2 V_2^* = 1$ (cf. [2, Lemma 4.1]). They are used to form K -theoretical addition in $\mathcal{M}(B) \cong M_2(\mathcal{M}(B))$. Note that we also write sometimes simply g for the action maps α_g, β_g, U_g etc.

In [11, Section 2] G -equivariant extensions of G -algebras A and B are introduced and identified with the set of G -equivariant $*$ -homomorphisms from A to $Q(B)$ $\text{Hom}_G(A, Q(B))$. The theory of this section goes essentially literally through. The [11, Theorem 2.1] can be ignored since it is only a continuity statement and hence trivial under our setting.

Theorem 2.7. *Discussion of [11, Theorem 2.2].*

Proof. To prove the G -equivariance of a certain $*$ -isomorphism $\kappa : E_1 \rightarrow E_2$ appearing in the diagram of [11, Theorem 2.2], we check first that it is E -equivariant. Let $p \in E$ and write $p^\perp = 1 - p$. Then by the diagram of [11, Theorem 2.2], $p^\perp \kappa p(E_1) \subseteq j_2(B)$ and $p \kappa^{-1} p^\perp(j_2(B)) = \{0\}$. Hence, $p \kappa^{-1} p^\perp \circ p^\perp \kappa p = 0$. Notice that $\kappa^{-1} p^\perp \kappa$ is a self-adjoint projection in $\mathcal{L}_{E_1}(E_1)$, and hence commutes with p , whence $p \kappa^{-1} p^\perp = 0$. Similarly $p^\perp \kappa^{-1} p = 0$ and so κ^{-1} intertwines p .

Let $g \in G$. We set then, as in Thomsen's proof, $\mu := \kappa^{-1} \circ \gamma_g^2 \circ \kappa \circ \gamma_{g^{-1}}^1 : gg^{-1} E_1 \rightarrow gg^{-1} E_1$ (also the maps j_1 and p_1 in the diagram of the proof of [11, Theorem 2.2] have to be restricted to $j_1 : gg^{-1} B \rightarrow gg^{-1} E_1$ and $p_1 : gg^{-1} E_1 \rightarrow gg^{-1} A$) and deduce that $\mu = \text{id}_{gg^{-1} E_1}$. Hence $\gamma_g^2 \circ \kappa = \kappa \circ \gamma_g^1$. \square

3. THE EQUIVARIANT EXTENSION GROUPS

Summing up [11, Section 3], for given G -algebras A and B an addition is introduced on the set $\text{Hom}_G(A, Q(B))$ (which, recall, encodes extensions) and equivariance relations are imposed on it. Two abelian extensions groups $\text{Ext}_G(A, B)$ and $\text{Ext}_G^h(A, B)$ come out, and it is shown that the canonical map $\text{Ext}_G(A, B) \rightarrow \text{Ext}_G^h(A, B)$ is surjective.

In this section we have to modify the definition of unitary equivalence as follows.

Definition 3.1. Two G -extensions $\varphi, \psi \in \text{Hom}_G(A, Q(B))$ are said to be *unitarily equivalent* if there exists a unitary $u \in \mathcal{M}(B)$ such that for all $g \in G$

1. $\widehat{\beta}_g(q_B(u)) = \widehat{\beta}_{gg^{-1}}(q_b(u))$,
2. $(\text{Ad } q_B(u)) \circ \varphi = \psi$.

Note that we made the only obvious modification in Definition 3.1 that $\widehat{\beta}_{gg^{-1}}$ was added in point 1. The construction of certain equivariant paths of isometries and unitaries in [11, Lemma 3.3] work also inverse semigroup equivariantly.

(Note that the unitary here has nothing to do with the fact that group actions are realized by unitaries.) In the statements and proofs of Lemma 3.2, Lemma 3.4 and Proposition 3.5 of Thomsen's paper the identities $\widehat{\beta}_g(q_B(X)) = q_B(X)$ and $\overline{\beta}_g(X) = X$, respectively, for $X = V, U, S, W, W_2, U_t, u$ have to be replaced by $\widehat{\beta}_g(q_B(X)) = \widehat{\beta}_{gg^{-1}}(q_B(X))$ and $\overline{\beta}_g(X) = \overline{\beta}_{gg^{-1}}(X)$, respectively. All these modifications are obvious. The extra terms $\widehat{\beta}_{gg^{-1}}$ and $\overline{\beta}_{gg^{-1}}$ may then slightly alter some computations, but since we shall demonstrate various similar computations in the next sections we omit presenting further details in this section.

4. AN APPROPRIATE PICTURE OF KK^1

Summarizing Section 4 of [11], the notion of the one-sided graded Kasparov group $KK_G^1(A, B) := KK^G(A, B \oplus B)$ is defined, and another group $\widetilde{KK}_G^1(A, B)$ of equivariant A, B - KK^1 -cocycles is introduced. An isomorphism $\widetilde{KK}_G^1(A, B) \cong KK_G^1(A, B)$ between both groups is then established.

In this section Thomsen recalls the definition of KK^G -theory, for which we use Definition 2.3. The definition of a Kasparov cycle $(\mathcal{E}, T) \in \mathbb{E}^G(A, B)$ requires that $(g(T) - Tgg^{-1}(1))a \in \mathcal{K}(\mathcal{E})$ for all $a \in A$ and $g \in G$. Thomsen remarks that we can drop here a at all and require the stronger version

$$(1) \quad g(T) - Tgg^{-1}(1) \in \mathcal{K}(\mathcal{E})$$

for all Kasparov cycles without changing KK^G -theory. This is true also in our case as pointed out in [2]. In Section 4 Thomsen introduces the notion of a unitary cocycle, for which we have to use the following modified version (as already pointed out in [2]).

Definition 4.1. Let (B, β) be a G -algebra. A *unitary β -cocycle* is a map $u : G \rightarrow \mathcal{M}(B)$ such that

$$\beta_{gg^{-1}} = u_g^* u_g, \quad u_{gg^{-1}} = u_g u_g^*, \quad u_{gh} = u_g \overline{\beta}_g(u_h)$$

for all $g, h \in G$.

This definition implies that u_g is a partial isometry with identical range and source projection. Furthermore, we have the identities

$$(2) \quad \beta_{gg^{-1}} = u_g^* u_g = u_g u_g^* = u_{gg^{-1}} \quad \text{and} \quad \overline{\beta}_g(u_{g^{-1}}) = u_g^*$$

for all $g \in G$, see [2, Lemma 3.3]. Thus, in our setting it would be more natural to simply speak about “ β -cocycles” instead of “unitary β -cocycles”. Informally we remark that a unitary β -cocycle will be used to combine it with the G -action $\overline{\beta}$ to form a new G -action $g \mapsto (\text{Ad } u_g) \circ \overline{\beta}_g$ on $\mathcal{M}(B)$. Cocycles come into play as ballast under some transformations of Kasparov cycles which are not G -equivariant, notably in [11, Theorem 4.3] as $u : G \rightarrow \mathcal{M}(B)$.

The definition of an equivariant A, B - KK^1 -cocycle changes slightly:

Definition 4.2. An *equivariant A, B - KK^1 -cocycle* is a triple (π, v, p) consisting of a $*$ -homomorphism $\pi : A \rightarrow \mathcal{M}(B)$, a unitary β -cocycle $v : G \rightarrow \mathcal{M}(B)$ (Definition 4.1) and a projection $p \in \mathcal{M}(B)$ such that

$$1. \quad (\text{Ad } v_g \circ \overline{\beta}_g)(\pi(a)) = \pi(\alpha_g(a)) \quad (\forall a \in A, g \in G)$$

2. $(\text{Ad } v_g \circ \overline{\beta}_g)(p) - \overline{\beta}_{gg^{-1}}(p) \in B \otimes \mathcal{K} \quad (\forall g \in G)$
3. $p\pi(a) - \pi(a)p \in B \otimes \mathcal{K} \quad (\forall a \in A).$

Only the additional operator $\overline{\beta}_{gg^{-1}}$ in the second point and the usage of the modified notion of β -cocycles have changed. These KK^1 -cocycles are also called triples in [11]. The definition of a degenerate cocycle has to be similarly slightly and obviously adapted as follows.

Definition 4.3. An equivariant A, B - KK^1 -cocycle (π, v, p) is called *degenerate* when

$$(3) \quad \begin{aligned} (\text{Ad } v_g \circ \overline{\beta}_g)(p) &= \overline{\beta}_{gg^{-1}}(p) \quad (\forall g \in G), \\ p\pi(\cdot) &= \pi(\cdot)p. \end{aligned}$$

The sum of KK^1 -cocycles is realized by any pair (V_1, V_2) of G -invariant isometries in $\mathcal{M}(B)$ such that $V_1 V_1^* + V_2 V_2^* = 1$.

Lemma 4.4. In [11, Lemma 4.2] Thomsen shows that the sum of an A, B - KK^1 -cocycle with a degenerate A, B - KK^1 -cocycle is homotopic to itself.

Proof. In accordance to (3), we only have to alter the line

$$\lim_{t \rightarrow 0} \text{Ad } [S_t u_g S_t^* + T_t v_g T_t^*] \circ \overline{\beta}_g(S_t q S_t^* + T_t p T_t^*) - (S_t q S_t^* + T_t p T_t^*)$$

in the proof of [11, Lemma 4.2] to

$$\lim_{t \rightarrow 0} \text{Ad } [S_t u_g S_t^* + T_t v_g T_t^*] \circ \overline{\beta}_g(S_t q S_t^* + T_t p T_t^*) - \overline{\beta}_{gg^{-1}}(S_t q S_t^* + T_t p T_t^*).$$

□

Lemma 4.5. After [11, Lemma 4.2], Thomsen defines a map $\Phi : \widetilde{KK}_G^1(A, B) \rightarrow KK_G^1(A, B)$ by setting $\Phi[\pi, v, p] = [E_B, \tilde{\pi}, S_p]$, where $E_B := B \oplus B$, $\tilde{\pi} = \pi \oplus \pi$ and $S_p(b_1, b_2) = ((2p - 1)b_1, (1 - 2p)b_2)$ for $b_1, b_2 \in B$. The Hilbert module E_B is endowed with the G -action $\tilde{v}_g := v_g \beta_g$.

Proof. Let us check that $[E_B, \tilde{\pi}, S_p]$ is a Kasparov cycle, where we only need to bother about the G -action. That $(E_B, \tilde{\pi})$ is a G -Hilbert bimodule we have already discussed in [2]. So let us only check identity $g(S_p) - S_p g(1) \in \mathcal{K}(E_B)$ from (1). We have, by considering only the first factor B of E_B ,

$$\begin{aligned} \tilde{v}_g S_p \tilde{v}_{g^{-1}}(b_1) &= v_g \beta_g((2p - 1)v_{g^{-1}} \beta_{g^{-1}}(b_1)) \\ &= 2v_g \overline{\beta}_g(p) \overline{\beta}_g(v_{g^{-1}}) \beta_{gg^{-1}}(b_1) - v_g \overline{\beta}_g(v_{g^{-1}}) \beta_{gg^{-1}}(b_1) \\ &= 2(\text{Ad } v_g \circ \overline{\beta}_g)(p) \beta_{gg^{-1}}(b_1) - v_g v_g^* \beta_{gg^{-1}}(b_1) \\ &\equiv 2p \beta_{gg^{-1}}(b_1) - \beta_{gg^{-1}}(b_1) = S_p \tilde{v}_g \tilde{v}_{g^{-1}}(b_1) \end{aligned}$$

for all $b_1 \in B$, because $\overline{\beta}_g(v_{g^{-1}}) = v_g^*$, $v_g v_g^* = \beta_{gg^{-1}}$ (see (2)), by Definition 4.2 and by using that $\overline{\beta}$ is the extension of β from B to $\mathcal{M}(B)$. □

The proof of [11, Theorem 4.3] is very similar to that of [10, Theorem 3.5] and was discussed inverse semigroup equivariantly in [2, Theorem 4.4], so we omit a rediscussion.

Lemma 4.6. We discuss [11, Lemma 4.4].

Proof. Thomsen notes that an application of Kasparov's [6, Lemma 6.1] yields a unitary in $T \in M_2(\mathcal{M}(B))$ which is G -invariant in Kasparov's case but only G -invariant modulo compacts in Thomsen's case. Similarly, when inspecting Kasparov's proof of his lemma (the G -action on $\mathcal{M}(B)$ is $v\beta$ and T is derived from the operator of a Kasparov cycle), in our case G -invariance of T means (according to Definition 2.3) that

$$\begin{pmatrix} v_g & 0 \\ 0 & w_g^2 \end{pmatrix} \overline{\beta_g \otimes id_{M_2}(T)} \begin{pmatrix} v_g^* & 0 \\ 0 & (w_g^2)^* \end{pmatrix} - T \begin{pmatrix} v_g \beta_{gg^{-1}} v_g^* & 0 \\ 0 & w_g^2 \beta_{gg^{-1}} (w_g^2)^* \end{pmatrix}$$

lies in $M_2(B)$. Recall that v_g and w_g^2 are partial isometries with source projections $v_g^* v_g = \beta_{gg^{-1}} = \overline{\beta}_g(1) = (w_g^2)^* w_g^2 \in \mathcal{M}(B)$ in the center of $\mathcal{M}(B)$ by (2). Hence we then also obtain easily relation "3." of [11, Lemma 4.4]. \square

5. TWISTED G -EXTENSIONS AND KK^1

The aim of [11, Section 5] is the introduction of a twisted extension group $\text{Ext}_{G,t}(A, B)$ and the establishment of an isomorphism $\text{Ext}_{G,t}(A, B) \cong \widetilde{KK}_G^1(A, B)$. Moreover, it is pointed out that two elements in $\text{Ext}_{G,t}(A, B)$ are identical if and only if their representatives are homotopic.

The notions of twisted extensions, unitary equivalence of twisted extensions, and degenerate twisted extensions need not be formally altered, excepting the implicitly self-evident fact that we have to use the modified notion of β -cycles of Definition 4.1. In the proofs of this section we have however slight adaption.

At first let us observe that unitary equivalence between two twisted extensions (φ, u) and (ψ, v) , where $\varphi, \psi : A \rightarrow Q(B)$ are G -equivariant $*$ -homomorphisms and u, v are β -cocycles, is an equivalence relation. Equivalence is defined through the existence of a unitary $u \in \mathcal{M}(B)$ such that $\text{Ad}_{q_B}(u) \circ \varphi = \psi$ and $v_g \overline{\beta}_g(u) - u u_g \in B$ for all $g \in G$. Let us demonstrate symmetry of the second relation. Taking the second relation for granted, we get

$$\begin{aligned} B &\ni \overline{\beta}_{g^{-1}}(v_g \overline{\beta}_g(u) - u u_g)^* = \overline{\beta}_{g^{-1}}(\overline{\beta}_g(u^*) \overline{\beta}_g(v_{g^{-1}}) - \overline{\beta}_g(u_{g^{-1}}) u^*) \\ &= u^* \overline{\beta}_{g^{-1}}(v_{g^{-1}}) - u_{g^{-1}} \overline{\beta}_{g^{-1}}(u^*) = u^* v_{g^{-1}} - u_{g^{-1}} \overline{\beta}_{g^{-1}}(u^*) \end{aligned}$$

by identities (2), the compatibility of the $\overline{\beta}$ -action, and the fact that $\overline{\beta}_{g^{-1}g}(v_{g^{-1}}) = \beta_{g^{-1}g} v_{g^{-1}} \beta_{g^{-1}g} = v_{g^{-1}}$ by $\beta_{g^{-1}g} = v_{g^{-1}} v_g^*$ by identities (2), as required.

Proposition 5.1. *Between [11, Lemma 5.1] and [11, Lemma 5.2], Thomsen shows that there is a group homomorphism $\Lambda : \text{Ext}_{G,t}((A, \alpha), (B, \beta)) \rightarrow \widetilde{KK}_G^1(A, B)$.*

Proof. Thomsen considers a twisted extension $(\varphi, u) \in \text{Ext}_{G,t}(A, B)$, where $\varphi : A \rightarrow Q(B)$ is a G -equivariant $*$ -homomorphism and u a β -cocycle. Then an inverse twisted extension $(\psi, v) \in \text{Ext}_{G,t}(A, B)$ to this one is chosen, that is, there is a degenerate twisted extension $(\lambda, w) \in \text{Ext}_{G,t}(A, B)$ such that $(\varphi + \psi, u + v)$ is unitarily equivalent to (λ, w) . One deduces that there exists a unitary $T \in \mathcal{M}(B)$ such that

$$\begin{aligned} \text{Ad}_{q_b}(T) \circ (\varphi + \psi) &= \lambda, \\ w_g \overline{\beta}_g(T) - T(V_1 u_g V_1^* + V_2 v_g V_2^*) &\in B \quad (\forall g \in G) \end{aligned}$$

in Thomsen's paper as well as here as the formal definitions around twisted extensions are unchanged. Then Thomsen states that one has

$$(4) \quad \begin{aligned} & \text{Ad } w_g \circ \overline{\beta}_g(TV_1V_1^*T^*) \\ &= \text{Ad } T \circ \text{Ad}(u+v)_g \circ \overline{\beta}_g(V_1V_1^*) = TV_1V_1^*T^* \quad (g \in G) \end{aligned}$$

in $\mathcal{M}(B)$ modulo B . This changes in our setting. We compute in $\mathcal{M}(B)$ modulo B

$$\begin{aligned} & \text{Ad } w_g \circ \overline{\beta}_g(TV_1V_1^*T^*) = w_g \overline{\beta}_g(T) V_1V_1^* \overline{\beta}_g(T)^* w_g^* \\ &= T(V_1u_gV_1^* + V_2v_gV_2^*) V_1V_1^* (V_1u_g^*V_1^* + V_2v_g^*V_2^*)T^* \\ &= TV_1u_gu_g^*V_1^*T^* = \beta_{gg^{-1}}TV_1V_1^*T^*\beta_{gg^{-1}} = \overline{\beta}_{gg^{-1}}(TV_1V_1^*T^*), \end{aligned}$$

because $\beta_{gg^{-1}} = u_gu_g^*$, see (2), is in the center of $\mathcal{M}(B)$. So we have an additional $\overline{\beta}_{gg^{-1}}$ in identity (4). Hence $(\overline{\lambda}, w, TV_1V_1^*T^*)$ is a KK^1 -cocycle in the sense of Definition 4.2, where $\overline{\lambda}: A \rightarrow \mathcal{M}(B)$ is so chosen that $\lambda = q_B \circ \overline{\lambda}$ and $\text{Ad } w_g \circ \overline{\beta}_g \circ \overline{\lambda} = \overline{\lambda} \circ \alpha_g$. This KK^1 -cocycle can be interpreted as an element in $\widetilde{KK}_G^1(A, B)$ (homotopy classes of $KK^1(A, B)$ -cocycles).

To see the well-definedness of the just constructed assignment $\text{Ext}_{G,t}(A, B) \rightarrow \widetilde{KK}_G^1(A, B)$, Thomsen considers another pair $(\psi^1, v^1), (\lambda^1, w^1) \in \text{Ext}_{G,t}(A, B)$ instead of $(\psi, v), (\lambda, w) \in \text{Ext}_{G,t}(A, B)$. To the other pair is associated a unitary $T^1 \in \mathcal{M}(B)$ instead of T . There are shown the existence of homotopies

$$(\overline{\lambda}^1, w^1, T^1V_1V_1^*T^{1*}) \sim (\mu, w^2, TV_1V_1^*T^*) \sim (\overline{\lambda}, w, TV_1V_1^*T^*),$$

where $S = TT^{1*}$, $\mu = \text{Ad } S \circ \overline{\lambda}^1$ and $w^2 = Sw_g^1\overline{\beta}_g(S^*) \in \mathcal{M}(B)$. Let us check that $(\mu, w^2, TV_1V_1^*T^*) \in \widetilde{KK}_G^1(A, B)$, so is a triple in the sense of Definition 4.2. Using that $\beta_{gg^{-1}}$ is in the center of $\mathcal{M}(B)$, we have

$$\begin{aligned} & \text{Ad } w_g^2 \circ \overline{\beta}_g \circ \mu(a) = \text{Ad } Sw_g^1\overline{\beta}_g(S^*) \circ \overline{\beta}_g \circ \text{Ad } S \circ \overline{\lambda}^1(a) \\ &= Sw_g^1\beta_gS^*\beta_{g^{-1}}\beta_gS\overline{\lambda}^1(a)S^*\beta_{g^{-1}}\beta_gS\beta_{g^{-1}}w_g^{1*}S^* = \mu(\alpha_g(a)), \text{ and} \\ & \text{Ad } w_g^2 \circ \overline{\beta}_g(TV_1V_1^*T^*) = Sw_g^1\beta_gS^*\beta_{g^{-1}}\beta_gTV_1V_1^*T^*\beta_{g^{-1}}\beta_gS\beta_{g^{-1}}w_g^{1*}S^* \\ &= TT^{1*}w_g^1\beta_gT^1T^*TV_1V_1^*T^*TT^{1*}\beta_{g^{-1}}w_g^{1*}T^1T^* \\ &\equiv TT^{1*}\overline{\beta}_{gg^{-1}}(T^1V_1V_1^*T^{1*})T^1T^* = \overline{\beta}_{gg^{-1}}(TV_1V_1^*T^*) \quad \text{mod } B \otimes \mathcal{K}. \end{aligned}$$

This proves the first two relations of Definition 4.2, and the third one is similar. The verification of the remaining parts is technically very similar to the last computations and thus we omit a further discussion. \square

[11, Lemma 5.2] goes through unchanged.

Theorem 5.2. *In [11, Theorem 5.3] it is remarked that the map $\Lambda: \text{Ext}_{G,t}((A, \alpha), (B, \beta)) \rightarrow \widetilde{KK}_G^1(A, B)$ is an isomorphism.*

Proof. The proof goes through unchanged. We only inspect the surjectivity proof, where from a given A, B - KK^1 -cocycle (π, v, p) a $*$ -homomorphism $\varphi: A \rightarrow Q(B)$ via $\varphi(a) = q_B(\pi(a)p)$ and then a twisted extension $(\varphi, v) \in \text{Ext}_{G,t}(A, B)$ is defined.

Checking that it is a twisted extension, we compute

$$\begin{aligned} \text{Ad } q_B(v_g) \circ \widehat{\beta}_g \circ \varphi(a) &= q_B(\text{Ad } v_g \overline{\beta}_g(\pi(a)p)) = q_B(v_g \overline{\beta}_g(\pi(a))v_g^* \overline{\beta}_{gg^{-1}}(p)) \\ &= q_B(v_g \overline{\beta}_g(\pi(a))v_g^*) q_B(p) = \varphi(\alpha_g(a)) \end{aligned}$$

by Definition 4.2 and $\beta_{gg^{-1}} = v_g v_g^*$ (identities (2)). \square

Also [11, Theorem 5.5] goes essentially through unchanged and involves only similar verifications and computations already having been demonstrated.

6. REMOVING THE TWIST

The aim of this section is the construction of a connection between twisted and untwisted extension groups, see Lemma 6.4 below. Here the paper [9] of Mingo and Phillips is essential and its adaption to inverse semigroups is presented in Section 8 as an appendix.

Lemma 6.1 (Cf. Lemma 6.1 of [11]). *We discuss here [11, Lemma 6.1], whose statement changes slightly. Obviously, in the statement of [11, Lemma 6.1] we have to use the identities*

$$\begin{aligned} \widehat{\beta}_g(q_B(u))\psi_1(a) &= \widehat{\beta_{gg^{-1}}}(q_B(u))\psi_1(a), \\ \widehat{\beta_g \otimes id_{M_4}}(q_{M_4(B)}(U_t)) &= \widehat{\beta_{gg^{-1} \otimes id_{M_4}}}(q_{M_4(B)}(U_t)) \end{aligned}$$

instead of the corresponding identities in Thomsen's paper without the appearances of the $\beta_{gg^{-1}}$ expressions.

Proof. In the proof of [11, Lemma 6.1] the identity [11, (6.1)] changes to

$$\widehat{\beta_g \otimes id_{M_2}}(q_D(S))\psi_1(a) = \widehat{\beta_{gg^{-1} \otimes id_{M_2}}}(q_D(S))\psi_1(a).$$

The function φ appearing in [11] takes the new form $\varphi(g) = \overline{\beta_g \otimes id_{M_2}}(S) - \overline{\beta_{gg^{-1} \otimes id_{M_2}}}(S)$. Thomsen then applies Kasparov's technical [7, Theorem 1.4], for which we use the corresponding and quite similar technical [3, Theorem 5.1] (the function ψ there has to be ignored).

In item "2." of the proof of [11, Lemma 6.1] we must replace the occurrences $\overline{\beta_g \otimes id_{M_2}}(Y)X - YX$ by

$$\overline{\beta_g \otimes id_{M_2}}(Y)X - \overline{\beta_{gg^{-1} \otimes id_{M_2}}}(Y)X$$

for $Y = S, SS$ and $X = x, xS$. The application of Kasparov's technical theorem yields positive elements $m, n \in \mathcal{M}(D)$ satisfying

$$\begin{aligned} \overline{\beta_g \otimes id_{M_2}}(m) - \overline{\beta_{gg^{-1} \otimes id_{M_2}}}(m), \overline{\beta_g \otimes id_{M_2}}(S)n - \overline{\beta_{gg^{-1} \otimes id_{M_2}}}(S)n, \\ n\overline{\beta_g \otimes id_{M_2}}(S) - n\overline{\beta_{gg^{-1} \otimes id_{M_2}}}(S) \in D \end{aligned}$$

rather than the corresponding relations without the appearance of $\overline{\beta_{gg^{-1} \otimes id_{M_2}}}$ as in Thomsen's paper. \square

Assume from now on that G is E -continuous!

For the definition of E -continuity see Definition 8.2 or Lemma 8.3. From now on, everywhere where Thomsen uses the G -Hilbert space $L^2(G)$ with its right regular representation we have to use the G -Hilbert $C_0(X)$ -module $\widehat{\ell}^2(G)$ defined in Definition 8.4 (there we need E -continuity). For simplicity, we shall thus also switch

to the notation $L^2(G) := \widehat{\ell^2}(G)$. The actions on $\mathcal{L}(L^2(G))$ and $\mathcal{K}(L^2(G))$ are the usual induced ones. For every G -Hilbert B -module or G -algebra \mathcal{E} , $L^2(G) \otimes \mathcal{E}$ in Thomsen [11] has to be replaced by the compatible tensor product $L^2(G) \otimes^X \mathcal{E}$ (cf. Definition 8.7), and likewise $\mathcal{K}(L^2(G)) \otimes \mathcal{E}$ in Thomsen [11] by $\mathcal{K}(L^2(G)) \otimes^X \mathcal{E}$ here. Every occurrence of the Hilbert space \mathbb{C} in [11] has to be substituted by the G -Hilbert $C_0(X)$ -module $C_0(X)$. This also includes, that we shall substitute the compact operators on a separable Hilbert space, \mathcal{K} , with its trivial G -action by the G -algebra $\mathcal{K}(C_0(X)^\infty)$ and write the G -algebra $\mathcal{K} \otimes A$ isomorphically as $\mathcal{K} \otimes A \cong \mathcal{K}(C_0(X)^\infty) \otimes^X A$ as G -algebras. (The isomorphism is of course given by $e_{ij} \otimes a \mapsto \theta_{1_{C_0(X)}, 1_{C_0(X)}}^{(i)} \otimes a$ (position i and j).)

Analogously as in Thomsen [11] one defines the G -algebra

$$\mathcal{K}_G := \mathcal{K}(C_0(X)^\infty) \otimes^X \mathcal{K}(L^2(G)).$$

For convenience of the reader, and since we are using a different $L^2(G)$ -space in our setting which requires higher attention if indeed everything works, we shall lay out more details and larger parts from Thomsen's proof than above in the rest of this section.

Lemma 6.2 (Cf. Lemma 6.2 of [11]). *Let $[(\varphi, u)] \in \text{Ext}_{G,t}(A, B)$ be a twisted extension. Write ρ for the G -action on \mathcal{K}_G . Then there is a unitary $w \in \mathcal{M}(B \otimes^X \mathcal{K}_G)$ such that*

$$(5) \quad w(u_g \otimes 1_{\mathcal{K}_G})(\beta_g \otimes \rho_g) = (\beta_g \otimes \rho_g)w.$$

Proof. Let the G -action on $L^2(G)$ (see Definition 8.4) be denoted by τ . We have two G -actions on $B^\infty \otimes^X L^2(G)$, namely $\mu_g := (u_g \beta_g)^\infty \otimes \tau_g$ and $\nu_g := (\beta_g)^\infty \otimes \tau_g$ for $g \in G$. The two corresponding G -Hilbert B -modules $B^\infty \otimes^X L^2(G)$ are G -equivariantly isomorphic by Lemma 8.8. Hence, there exists a unitary $v \in \mathcal{L}_B(B^\infty \otimes^X L^2(G))$ intertwining the aforementioned two G -actions. Consider the isomorphism of Hilbert B -modules, see Kasparov [5, Theorem 2.1],

$$\gamma : \mathcal{L}_B(L^2(G) \otimes^X B^\infty) \rightarrow \mathcal{M}(\mathcal{K}(L^2(G) \otimes^X B^\infty)) : \gamma(T) = (T_1, T_2),$$

where (T_1, T_2) is the double centralizer given by $T_1(\theta_{x,y}) = \theta_{T_x,y}$ and $T_2(\theta_{x,y}) = \theta_{x,T^*y}$ for $x, y \in L^2(G) \otimes^X B^\infty$.

Note that γ is G -equivariant for both actions induced by μ and ν , respectively, on the domain of γ , and their corresponding induced actions on the range of γ , respectively, since

$$(g(T))_1(\theta_{x,y}) = \theta_{gTg^{-1}x,y} = \theta_{gTg^{-1}x,gg^{-1}y} = g \circ T_1 \circ g^{-1}(\theta_{x,y})$$

by compatibility of the inner product (i.e. $gg^{-1}\langle x, y \rangle = \langle gg^{-1}x, y \rangle$) and the identity

$$g(\theta_{x,y})(z) = g(x\langle y, g^{-1}(z) \rangle) = \theta_{gx,gy}(z)$$

for all $g \in G$ and $x, y, z \in L^2(G) \otimes^X B^\infty$. Notice that

$$\begin{aligned} \mathcal{K}(L^2(G) \otimes^X B^\infty) &\cong \mathcal{K}(L^2(G)) \otimes^X \mathcal{K}(B^\infty) \cong \mathcal{K}(L^2(G)) \otimes^X (\mathcal{K} \otimes B) \\ &\cong \mathcal{K}(L^2(G)) \otimes^X (\mathcal{K}(C_0(X)^\infty) \otimes^X B) \cong \mathcal{K}_G \otimes^X B \end{aligned}$$

isomorphically as G -algebras by P.-Y. Le Gall [8, Proposition 4.2.(a)]. We may thus identify the range of γ with $\mathcal{M}(B \otimes^X \mathcal{K}_G)$.

Set $w := \gamma(v)$. Recall that $v\mu_g = \nu_g v$. Hence $\nu_g v \nu_{g^{-1}} = v\mu_g \nu_{g^{-1}}$. Since $\mu_g \nu_{g^{-1}} = (u_g \otimes 1)(\beta_{gg^{-1}} \otimes \rho_{gg^{-1}})$, we obtain from $\gamma(\nu_g v \nu_{g^{-1}}) = \gamma(v)\gamma(\mu_g \nu_{g^{-1}})$ that

$$(\beta_g \otimes \rho_g)w(\beta_{g^{-1}} \otimes \rho_{g^{-1}}) = w(u_g \beta_{gg^{-1}} \otimes \rho_{gg^{-1}}).$$

Multiplying from right with $\beta_g \otimes \rho_g$, and noting that $\beta_{g^{-1}g} \otimes \rho_{g^{-1}g}$ is in the center, yields the claim. \square

Definition 6.3 (Cf. [11] after Lemma 6.2). For a G -algebra (C, γ) and a homomorphism $\varphi \in \text{Hom}_G(A, Q(B))$ we denote by $\varphi \hat{\otimes} id_C \in \text{Hom}_G(A \otimes^X C, Q(B \otimes^X C))$ the canonical map $\varphi \hat{\otimes} id_C(a \otimes c) = q_B^{-1}(\varphi(a)) \otimes c + B \otimes^X C$.

Lemma 6.4 (Cf. [11] after Lemma 6.2). For every twisted extension $[(\varphi, u)] \in \text{Ext}_{G,t}(A, B)$ choose a unitary $w \in \mathcal{M}(B \otimes^X \mathcal{K}_B)$ according to Lemma 6.2. Then there exists a homomorphism

$$\Theta : \text{Ext}_{G,t}((A, \alpha), (B, \beta)) \rightarrow \text{Ext}_G(A \otimes^X \mathcal{K}_G, Q(B \otimes^X \mathcal{K}_G))$$

defined by $\Theta([\varphi, u]) = [Ad_{q_{B \otimes^X \mathcal{K}_G}}(w) \circ (\varphi \hat{\otimes} id_{\mathcal{K}_G})]$.

Proof. Recall from Section 5 of [11] that $\text{Ext}_{G,t}$ consists of twisted extension divided by the equivalence relation \sim that two twisted extensions e_1, e_2 are equivalent if adding both with two (possibly different) degenerate twisted extensions, then they are unitarily equivalent, and finally considering in this quotient space only invertible extensions with respect to addition (direct sum of extensions). The extension group Ext_G is defined analogously, without the word “twisted” everywhere.

At first we are going to prove that $\Theta(\varphi, u)$ is indeed in Ext_G . For $T \in \mathcal{M}(B \otimes^X \mathcal{K}_G)$ we have

$$\begin{aligned} & \text{Ad}(w) \circ \text{Ad}(u_g \otimes 1_{\mathcal{K}_G}) \circ \overline{\beta_g \otimes \rho_g}(T) \\ &= w(u_g \otimes 1_{\mathcal{K}_G})(\beta_g \otimes \rho_g)T(\beta_{g^{-1}} \otimes \rho_{g^{-1}})(u_g^* \otimes 1_{\mathcal{K}_G})w^* \\ &= (\beta_g \otimes \rho_g)wTw^*(u_{g^{-1}} \otimes 1_{\mathcal{K}_G})(\beta_{g^{-1}} \otimes \rho_{g^{-1}})w^* \\ &= (\beta_g \otimes \rho_g)wTw^*(\beta_{g^{-1}} \otimes \rho_{g^{-1}}) \\ (6) \quad &= \overline{\beta_g \otimes \rho_g} \circ \text{Ad}(w)(T) \end{aligned}$$

in $\mathcal{M}(B \otimes^X \mathcal{K}_G)$ by Lemma 6.2 and (2). Since (φ, u) is a twisted extension, and so $Ad_{q_B}(u_g) \circ \widehat{\beta_g} \circ \varphi(a) = \varphi(\alpha_g(a))$, see [11, Section 5], we get

$$Ad_{q_{B \otimes^X \mathcal{K}_G}}(u_g \otimes 1_{\mathcal{K}_G}) \circ \widehat{\beta_g \otimes \rho_g} \circ (\varphi \hat{\otimes} id_{\mathcal{K}_G}) = (\varphi \hat{\otimes} id_{\mathcal{K}_G}) \circ (\alpha_g \otimes \rho_g).$$

Applying $Ad_{q_{B \otimes^X \mathcal{K}_G}}(w)$ on this identity shows with identity (6) that

$$(7) \quad Ad_{q_{B \otimes^X \mathcal{K}_G}}(w) \circ (\varphi \hat{\otimes} id_{\mathcal{K}_G}) \in \text{Hom}_G(A \otimes^X \mathcal{K}_G, Q(B \otimes^X \mathcal{K}_G)),$$

in other words, this map is G -equivariant. Hence, $\Theta(\varphi, u)$ is in Ext_G .

If $(\hat{\varphi}, u)$ is a degenerate twisted extension, that is, $Ad(u_g) \circ \widehat{\beta_g} \circ \hat{\varphi}(a) = \hat{\varphi}(\alpha_g(a))$ with $\hat{\varphi} \in \text{Hom}_G(A, \mathcal{M}(B))$, see [11, Section 5], then similar as above we achieve

$$Ad(w) \circ (\hat{\varphi} \hat{\otimes} id_{\mathcal{K}_G}) \in \text{Hom}_G(A \otimes^X \mathcal{K}_G, \mathcal{M}(B \otimes^X \mathcal{K}_G)).$$

In other words, if $(\hat{\varphi}, u)$ is degenerate, so is $\Theta(\hat{\varphi}, u)$.

Our next aim is to show that Θ respects unitary equivalence. Assume that two twisted extensions (φ, u) and (ψ, v) are unitarily equivalent, $(\varphi, u) \cong (\psi, v)$, see [11, Section 5]. That means, that there exists a unitary $s \in \mathcal{M}(B)$ such that

$$(8) \quad \text{Ad } q_B(s) \circ \varphi = \psi, \quad v_g \overline{\beta}_g(s) - s u_g \in B$$

for all $g \in G$. Let $t \in \mathcal{M}(B \otimes^X \mathcal{K}_G)$ denote the unitary w for the twisted extension (ψ, v) from Lemma 6.2. From (8) it follows

$$(9) \quad \begin{aligned} & \text{Ad } q_{B \otimes^X \mathcal{K}_G}(t(s \otimes 1_{\mathcal{K}_G})w^*) \circ \text{Ad } q_{B \otimes^X \mathcal{K}_G}(w) \circ (\varphi \hat{\otimes} \text{id}_{\mathcal{K}_G}) \\ &= \text{Ad } q_{B \otimes^X \mathcal{K}_G}(t) \circ (\psi \hat{\otimes} \text{id}_{\mathcal{K}_G}). \end{aligned}$$

Moreover, computing $\overline{\beta_g \otimes \rho_g}(w^*) = (\beta_{gg^{-1}} \otimes \rho_{gg^{-1}})(u_g^* \otimes 1)w^*$ by multiplying (5) from the right with $\beta_{g^{-1}} \otimes \rho_{g^{-1}}$ and taking the adjoint, and similarly computing $\overline{\beta_g \otimes \rho_g}(t) = t(v_g \otimes 1)(\beta_{gg^{-1}} \otimes \rho_{gg^{-1}})$, we have for all $m \in \mathcal{M}(B)$ and $k \in \mathcal{K}_G$ (in $\mathcal{M}(B \otimes^X \mathcal{K}_G)$)

$$\begin{aligned} & \left(\overline{\beta_g \otimes \rho_g}(t(s \otimes 1)w^*) - \overline{\beta_{gg^{-1}} \otimes \rho_{gg^{-1}}}(t(s \otimes 1)w^*) \right) w(m \otimes k)w^* \\ &= \left(t(v_g \otimes 1) \overline{\beta_g \otimes \rho_g}(s \otimes 1)(u_g^* \otimes 1)w^* - t(\overline{\beta_{gg^{-1}}}(s) \otimes \overline{\rho_{gg^{-1}}}(1))w^* \right) w(m \otimes k)w^* \\ &= t \left((v_g \overline{\beta}_g(s) u_g^* m - \overline{\beta}_{gg^{-1}}(s) m) \otimes \overline{\rho}_g(1)k \right) w^* \in B \otimes^X \mathcal{K}_G \end{aligned}$$

by the fact that $v_g \overline{\beta}_g(s) u_g^* - \beta_{gg^{-1}}(s) \in B$ by (8) and (2).

The last computation shows that

$$(10) \quad \begin{aligned} & \widehat{\beta_g \otimes \rho_g} \left(q_{B \otimes^X \mathcal{K}_G}(t(s \otimes 1)w^*) \right) \text{Ad } q_{B \otimes^X \mathcal{K}_G}(w) \circ (\varphi \hat{\otimes} \text{id}_{\mathcal{K}_G}) \\ &= \widehat{\beta_{gg^{-1}} \otimes \rho_{gg^{-1}}} \left(q_{B \otimes^X \mathcal{K}_G}(t(s \otimes 1)w^*) \right) \text{Ad } q_{B \otimes^X \mathcal{K}_G}(w) \circ (\varphi \hat{\otimes} \text{id}_{\mathcal{K}_G}). \end{aligned}$$

An application of Lemma 6.1 to identities (9) and (10) (recall (7)) shows that $\Theta(\varphi, u) + 0$ (sum operator, cf. [11, Lemma 3.1]) and $0 + \Theta(\psi, v)$ are unitarily equivalent in the sense of Definition 3.1.

We leave the verification of the last claim, that Θ respects direct sums, and so also invertible elements go to invertible elements, to the reader by verbatim following the corresponding proof in the last part of [11, Section 6]. \square

7. EQUIVARIANT KK -THEORY AND C^* -EXTENSIONS

The aim of [11, Section 7] is the proof of the main result, the identification of extension groups with Kasparov groups, see Theorems 7.1 and 7.2.

Again, in this section we must replace the tensor products $A \otimes \mathcal{K}_G$ and so on by the balanced tensor products $A \otimes^X \mathcal{K}_G$ and so forth. For the discussion in [11, Section 7.2] we remark that \mathcal{K}_G is equivalent to $C_0(X)$ in KK^G . This was proved in [1, Proposition 5.14]. (Note that $\mathcal{K}(L^2(G))$ is equivalent to $\mathcal{K}(L^2(G) \oplus C_0(X))$ which is equivalent to $\mathcal{K}(C_0(X))$ in KK^G .)

We begin by restating the main results of Thomsen's paper, slightly adapted to our setting. These are, of course, also the main results of this note. Recall from [11, Section 4] that $KK_G^1(A, B) := KK^G(A, B \oplus B)$, where A and B are trivially graded and $B \oplus B$ is obviously graded by the flip operator.

Theorem 7.1 (Cf. Theorem 7.1 of [11]). *Assume that G is a countable, E -continuous inverse semigroup. We may identify the KK_G^1 -groups with the extension groups of the stabilized G -algebras. More precisely, for all ungraded separable G -algebras A and B , where B is stable, we have*

$$\begin{aligned} KK_G^1(A, B) &\cong \text{Ext}_G(A \otimes^X \mathcal{K}_G, B \otimes^X \mathcal{K}_G) \cong \text{Ext}_G^h(A \otimes^X \mathcal{K}_G, B \otimes^X \mathcal{K}_G) \\ &\cong \text{Ext}_{G,t}(A, B) \cong \widetilde{KK}_G^1(A, B). \end{aligned}$$

The last two isomorphisms are a restatement of [11, Theorem 4.3] and [11, Theorem 5.3]. Of course, the first two isomorphisms hold also for unstable B . The following theorem just states that the second isomorphism is canonical.

Theorem 7.2 (Cf. Theorem 7.2 of [11]). *Assume that G is E -continuous. Let $\varphi, \psi \in \text{Hom}_G(A \otimes^X \mathcal{K}_G, Q(B \otimes^X \mathcal{K}_G))$ be invertible G -extensions. Then $[\varphi] = [\psi]$ in $\text{Ext}_G(A \otimes^X \mathcal{K}_G, B \otimes^X \mathcal{K}_G)$ if and only if φ and ψ are homotopic, that is, there is an invertible extension $\Phi \in \text{Hom}_G(A \otimes^X \mathcal{K}_G, Q(IB \otimes^X \mathcal{K}_G))$ such that $\widehat{\pi_0} \circ \Phi = \varphi$ and $\widehat{\pi_1} \circ \Phi = \psi$.*

Lemma 7.3. [11, Lemma 7.3] *holds verbatim. (Recall that an operator $u \in \mathcal{M}((A, \alpha))$ is $\overline{\alpha}$ -invariant if $\alpha_g \circ u = u \circ \alpha_g$ for all $g \in G$.)*

Proof. In [11, Lemma 7.3] we have to alter two lines in its proof. We have to take

$$\beta_g \widehat{\otimes id_{M_2}} \left(\begin{pmatrix} \overline{\psi}(U) & 0 \\ 0 & \overline{\psi}(U^*) \end{pmatrix} \right) x = \beta_{gg^{-1}} \widehat{\otimes id_{M_2}} \left(\begin{pmatrix} \overline{\psi}(U) & 0 \\ 0 & \overline{\psi}(U^*) \end{pmatrix} \right) x$$

and

$$\beta_g \widehat{\otimes id_{M_2}} (q_{B \otimes M_2}(W)) \begin{pmatrix} \overline{\psi}(a) & 0 \\ 0 & 0 \end{pmatrix} = \beta_{gg^{-1}} \widehat{\otimes id_{M_2}} (q_{B \otimes M_2}(W)) \begin{pmatrix} \overline{\psi}(a) & 0 \\ 0 & 0 \end{pmatrix}$$

instead of the corresponding identities without appearance of $\beta_{gg^{-1}} \widehat{\otimes id_{M_2}}$ in Thomsen's paper. Of course, instead of [11, Lemma 6.1] we have to apply Lemma 6.1 in the proof. \square

Most of the remainder of Section 7 of Thomsen's paper after [11, Lemma 7.3] is dedicated to the following lemma stated at the beginning of [11, Section 7.3].

Lemma 7.4. *There exists a well-defined, injective map*

$$\begin{aligned} \kappa : \text{Hom}_G(A \otimes^X \mathcal{K}_G, Q(B \otimes^X \mathcal{K}_G)) / \sim \\ \rightarrow \text{Hom}_G(A \otimes^X \mathcal{K}_G \otimes^X \mathcal{K}_G, Q(B \otimes^X \mathcal{K}_G \otimes^X \mathcal{K}_G)) / \sim \end{aligned}$$

given by $[\psi] \mapsto [\psi \hat{\otimes} id_{\mathcal{K}_G}]$, where \sim denotes the equivalence relation on $\text{Hom}_G(A, Q(B))$ whose quotient defines the set of invertible elements called $\text{Ext}_G(A, B)$.

Proof. As in Thomsen [11] we remark that κ is similarly defined like Θ of Lemma 6.3 and so the proof that κ is well-defined is similar as the corresponding proof there by an application of Lemma 6.1.

After [11, Lemma 7.3] Thomsen considers the G -equivariant Hilbert space $\mathbb{C} \oplus L^2(G)$ which in our case turns to the G -Hilbert $C_0(X)$ -module $C_0(X) \oplus L^2(G)$ with

diagonal G -action $\tau_g^+(\lambda, \psi) := g(\lambda) \oplus g(\psi)$ for all $g \in G, \lambda \in C_0(X)$ and $\psi \in L^2(G)$. This induces a canonical G -action on the C^* -algebra

$$\mathcal{K}_G^+ := \mathcal{K}(C_0(X)^\infty) \otimes^X \mathcal{K}(C_0(X) \oplus L^2(G)).$$

We proceed then as in Thomsen's paper after [11, Lemma 7.3]. Thomsen considers a similar map, κ^+ say, which replaces the new copies of \mathcal{K}_G in the image of κ by \mathcal{K}_G^+ and maps $[\psi]$ to $[\psi \hat{\otimes} \mathcal{K}_G^+]$. A G -invariant projection $e \in \mathcal{K}_G^+$ is chosen for which we take $e := e_{11} \otimes e_{11} \in \mathcal{K}_G^+$ where $e_{11} = \theta_{1_{C_0(X)}, 1_{C_0(X)}} \in \mathcal{K}(C_0(X) \oplus L^2(G))$, $\mathcal{K}(C_0(X)^\infty)$ denotes the corner projection. By Corollary 8.10 we may choose a G -invariant isometry $V \in \mathcal{M}(B \otimes^X \mathcal{K}_G \otimes^X \mathcal{K}_G^+)$ such that $VV^* = 1_{\mathcal{M}(B \otimes^X \mathcal{K}_G)} \otimes e$. We can then literally proceed as in Thomsen's paper to show that κ^+ is injective. Also the remaining proof where it is shown that κ is injective goes verbatim through in our setting (with only obvious adaption as taking $A \otimes^X \mathcal{K}_G$ rather than $A \otimes \mathcal{K}_G$). A certain G -invariant isometry $W \in \mathcal{M}(A \otimes^X \mathcal{K}_G \otimes^X \mathcal{K}_G^+)$ with range projection $1_{\mathcal{M}(A \otimes^X \mathcal{K}_G \otimes^X \mathcal{K}_G)}$ is then again chosen by Corollary 8.10. \square

8. ADAPTION OF A PAPER OF MINGO AND PHILLIPS

In this section we introduce the *compatible* $\ell^2(G)$ -space $\widehat{\ell}^2(G)$ for an inverse semigroup G which replaces the corresponding classical space for groups with its regular representation. This new notion is necessary for $\widehat{\ell}^2(G)$ to become a G -Hilbert $C_0(X)$ -module, and so particularly $\mathcal{L}(\ell^2(G))$ a G -algebra. The classical $\ell^2(G)$ -space would not become a G -Hilbert \mathbb{C} -module. Moreover, we restate some central results from the paper of Mingo and Phillips [9] for this new ℓ^2 -space. All from this section is actually taken from [1].

Definition 8.1. Define X to be the locally compact Hausdorff space and Gelfand spectrum of the commutative C^* -algebra $C^*(E)$ which is freely generated by the commuting, self-adjoint projections $e \in E$. That is, $C_0(X) \cong C^*(E)$ via $1_e \leftrightarrow e$ for all $e \in E$ (so we identify in our notation the projection e with its carrier set in X). We turn this C^* -algebra to a G -algebra under the G -action $g(1_e) := 1_{ge}g^{-1}$ for all $g \in G$ and $e \in E$.

In the next few paragraphs (until Lemma 8.5) we shall identify elements $e \in E$ with its characteristic function 1_e in $C_0(X)$. Write $\text{Alg}^*(E)$ for the dense $*$ -subalgebra of $C_0(X)$ generated by the characteristic functions 1_e for all $e \in E$. Moreover, write $\bigvee_i f_i : X \rightarrow \mathbb{C}$ for the pointwise supremum of a family of functions $f_i : X \rightarrow \mathbb{C}$. We shall use the well-known order relation on G defined by $g \leq h$ iff $g = eh$ for some $e \in E$.

Definition 8.2. An inverse semigroup G is called *E-continuous* if the function $\bigvee\{e \in E \mid e \leq g\} \in \mathbb{C}^X$ (in precise notation: $\bigvee\{1_e \in C_0(X) \mid e \in E, e \leq g\} \in \mathbb{C}^X$) is a *continuous* function in $C_0(X)$ for all $g \in G$.

Lemma 8.3. An inverse semigroup G is *E-continuous* if and only if for every $g \in G$ there exists a finite subset $F \subseteq E$ such that $\bigvee\{e \in E \mid e \leq g\} = \bigvee\{e \in F \mid e \leq g\}$.

Proof. If $\bigvee\{e \in E \mid e \leq g\} = 1_K \in C_0(X)$ for a clopen subset $K \subseteq X$ then K must be compact. Hence $K = \bigcup\{\text{carrier}(1_e) \subseteq X \mid e \in E, e \leq g\}$ allows a finite subcovering. \square

Definition 8.4 (Compatible $L^2(G)$ -space). Let G be an E -continuous inverse semi-group. Write c for the linear span of all functions $\varphi_g : G \rightarrow \mathbb{C}$ (in the linear space \mathbb{C}^G) defined by

$$\varphi_g(t) := 1_{\{t \leq g\}}$$

(characteristic function) for all $g, t \in G$. Endow c with the G -action $g(\varphi_h) := \varphi_{gh}$ for all $g, h \in G$. Turn c to an $\text{Alg}^*(E)$ -module by setting $\xi e := e(\xi)$ for all $\xi \in c$ and $e \in E$. Define an $\text{Alg}^*(E)$ -valued inner product on c by

$$(11) \quad \langle \varphi_g, \varphi_h \rangle := \bigvee \{e \in E \mid eg = eh, e \leq gg^{-1}hh^{-1}\}.$$

The norm completion of c is a G -Hilbert $C_0(X)$ -module denoted by $\widehat{\ell^2}(G)$.

We discuss the last definition. At first notice that $\langle \varphi_g, \varphi_h \rangle = gg^{-1} \bigvee \{e \in E \mid e = ehg^{-1}\}$ (observe that $e = ehg^{-1}$ implies $e \leq hg^{-1}gh^{-1}$), so that by E -continuity $\langle \varphi_g, \varphi_h \rangle$ is in $C_0(X)$ and actually even in $\text{Alg}^*(E)$ by Lemma 8.3, and $e \in E$ in (11) can be replaced by $e \in F$ for some finite subset $F \subseteq E$. The identities $\langle \varphi_g, \varphi_h \rangle = \langle \varphi_h, \varphi_g \rangle$, $\langle \varphi_g, \varphi_h f \rangle = \langle \varphi_g f, \varphi_h \rangle = \langle \varphi_g, \varphi_h \rangle f$, $j(\langle \varphi_g, \varphi_h \rangle) = \langle j(\varphi_g), j(\varphi_h) \rangle$ for all $g, h, j \in G$ and $f \in E$ are easy to check. We note that (11) is positive definite. Indeed, assume $\langle x, x \rangle = 0$ for $x = \sum_{i=1}^n \lambda_i \varphi_{g_i}$ with nonzero $\lambda_i \in \mathbb{C}$ and $g_i \in G$ mutually different. Choose g_j such that no other g_i satisfies $g_j g_j^{-1} < g_i g_i^{-1}$. Hence, $\langle \varphi_{g_j}, \varphi_{g_j} \rangle = g_j g_j^{-1}$ but $\langle \varphi_{g_i}, \varphi_{g_k} \rangle \neq g_j g_j^{-1}$ for all combinations where $i \neq k$. By linear independence of the projections E in $\text{Alg}^*(E)$ λ_j must be zero; contradiction. The last proof also shows the following lemma.

Lemma 8.5. *The vectors $(\varphi_g)_{g \in G} \subseteq \widehat{\ell^2}(G)$ are linearly independent.*

Every G -algebra (A, α) is endowed with a $*$ -homomorphism $\pi : C_0(X) \cong C^*(E) \rightarrow Z\mathcal{M}(A)$ given by $\pi(e)(a) = \alpha_e(a)$ for all $e \in E$ and $a \in A$. If G has a unit then A is a $C_0(X)$ -algebra in the sense of Kasparov [7, Definition 1.5]; but this extra property is unimportant in this note.

Definition 8.6. We define (cf. Kasparov [7, Definition 1.6]) the balanced tensor product $A \otimes^X B$ of two G -algebras A and B as the quotient of $A \otimes_{\max} B$ divided by the relations $e(a) \otimes b \equiv a \otimes e(b)$ for all $e \in E, a \in A$ and $b \in B$.

Definition 8.7. Let \mathcal{E} be a G -Hilbert B -module. Then $L^2(G, \mathcal{E}) := \widehat{\ell^2}(G) \otimes^X \mathcal{E}$ is a G -Hilbert B -module, where \otimes^X denotes the $C_0(X)$ -balanced exterior tensor product as defined by Le Gall [8, Definition 4.2] (or in this case equivalently, the internal tensor product $\otimes_{C_0(X)}$).

Lemma 8.8 (Cf. Lemma 2.3 of [9]). *If \mathcal{E}_1 and \mathcal{E}_2 are G -Hilbert A -modules which are isomorphic as Hilbert A -modules then $L^2(G, \mathcal{E}_1)$ and $L^2(G, \mathcal{E}_2)$ are isomorphic as G -Hilbert A -modules.*

Proof. Let $u \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ be a unitary operator. Note that $gg^{-1} \in \mathcal{L}(\mathcal{E}_i)$ commutes with u for all $g \in G$ since u is A -linear and $gg^{-1}(\xi)a = \xi gg^{-1}(a)$ for all $\xi \in \mathcal{E}_i, a \in A$. Then it can be checked that $V : L^2(G, \mathcal{E}_1) \rightarrow L^2(G, \mathcal{E}_2)$ given by $V(\varphi_g \otimes \xi) := \varphi_g \otimes gug^{-1}(\xi)$ defines an isomorphism of G -Hilbert A -modules. We

show that V is G -equivariant. For $h \in G$ we have

$$\begin{aligned} h(V(\varphi_g \otimes \xi)) &= h\varphi_g \otimes hgug^{-1}h^{-1}h(\xi) \\ &= \varphi_{hg} \otimes hg(u)(h(\xi)) \\ &= V(h(\varphi_g \otimes \xi)), \end{aligned}$$

because $h^{-1}h \in \mathcal{L}(\mathcal{E}_i)$ commutes with $gug^{-1} \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$.

For the inner product note that $\langle \varphi_g, \varphi_h \rangle = \sum_{f \in F} f$ for a finite set $F \subseteq E$ with $fg = fh$ and $f \leq gg^{-1}hh^{-1}$ by Lemma 8.3, so that

$$\begin{aligned} \langle V(\varphi_g \otimes \xi), V(\varphi_h \otimes \eta) \rangle &= \sum_{f \in F} f \otimes \langle fgug^{-1}f(\xi), fhuh^{-1}f(\eta) \rangle \\ &= \langle \varphi_g \otimes \xi, \varphi_h \otimes \eta \rangle. \end{aligned}$$

□

Corollary 8.9 (Cf. Theorem 2.4 of [9]). *Let \mathcal{E} be a G -Hilbert A -module which is countably generated and full as a Hilbert A -module. Then $L^2(G, \mathcal{E})^\infty$ is isomorphic to $L^2(G, A)^\infty$ by a G -equivariant isomorphism of Hilbert A -modules.*

Proof. Same proof as in Mingo and Phillips [9], Theorem 2.4, but by applying Lemma 8.8 instead of [9, Lemma 2.3]. □

Corollary 8.10 (Cf. Corollary 2.6 of [9]). *Let (A, α) be a G -algebra and suppose that A has a strictly positive element. If $p \in \mathcal{M}(A)$ is a full G -invariant projection then $p \otimes 1_{\mathcal{M}(\mathcal{K}_G)} \sim 1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(\mathcal{K}_G)}$ (Murray-von Neumann equivalence) in $\mathcal{M}(A \otimes^X \mathcal{K}_G)$ by a G -invariant partial isometry.*

Proof. The proof of the original goes verbatim through. The usage of the balanced tensor product \otimes^X instead of \otimes is obligatory. □

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